University of California San Diego Fall 2021

ECE 259A: Midterm Exam

Instructions: There are four problems, weighted as shown below. The exam is open book and open notes: you may use any auxiliary material that you like as long as it is on paper.

Good luck!

Problem 1. (25 points)

Let $\mathbb{C}$ be the binary linear code of length $n = 10$, generated by the following matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

a. Express G in systematic form, and compute a parity-check matrix for this code.

b. What is the minimum distance of $\mathbb{C}^{\perp}$, the dual code of $\mathbb{C}$?

c. Compute the syndrome of the vector $\underline{y} = (1,0,1,1,0,0,1,1,0,0)$ with respect to the parity-check matrix found in part (**a**).

d. If the vector $\underline{y} = (1,0,1,1,0,0,1,1,0,0)$ from part (**c**) is observed at the output of a binary symmetric channel, what is the most likely transmitted codeword of $\mathbb{C}$?

Hint: Compare the syndrome from part (**c**) with the columns of the parity-check matrix.

Problem 2. (20 points)

A binary code $\mathbb{C}$ is said to be *doubly-even* if the weight of all the codewords of $\mathbb{C}$ is divisible by 4. Suppose that a binary linear code $\mathbb{C}$ and its dual $\mathbb{C}^{\perp}$ are both doubly-even. Prove that $\mathbb{C} = \mathbb{C}^{\perp}$.

Problem 3. (25 points)

Let $\mathcal{E}$ be an arbitrary subset of $\mathbb{F}_2^n$ of cardinality $|\mathcal{E}| = M$, such that $\underline{0} \in \mathcal{E}$. We say that a code $\mathbb{C} \subset \mathbb{F}_2^n$ detects all error patterns in the set $\mathcal{E}$ if $\underline{c}_1 + \underline{e} \neq \underline{c}_2$ for all $\underline{e} \in \mathcal{E}$ and for all distinct $\underline{c}_1, \underline{c}_2 \in \mathbb{C}$ (or, equivalently, if $(\underline{c} + \mathcal{E}) \cap \mathbb{C} = \{\underline{c}\}$ for all $\underline{c} \in \mathbb{C}$).

Show that there exist codes of cardinality $|\mathbb{C}| \geqslant 2^n / M$ that detect all error patterns in the set $\mathcal{E}$.

Problem 4. (30 points)

Let $\mathbb{C}$ be an (n,k,d) linear code over the field $\mathrm{GF}(q) = \{0, \alpha^0, \alpha^1, \ldots, \alpha^{q-2}\}$, where $q \geqslant 3$. It is known that $\mathbb{C}$ is at the same time an MDS code and a non-trivial perfect code.

a. What are the parameters n,k,d of $\mathbb{C}$ (in terms of q)?

b. Write down a parity-check matrix for $\mathbb{C}$.

Hint: Use the fact that the only non-trivial perfect linear codes over a field are the binary Golay code over $\mathrm{GF}(2)$, the ternary Golay code over $\mathrm{GF}(3)$, and the Hamming codes over $\mathrm{GF}(q)$ for all q.